Redundant Manipulators

$$\dot{x} = \begin{pmatrix} V \\ \omega \end{pmatrix} = J \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} \qquad n > 6$$

$6 \times 1 \qquad 6 \times n \qquad n \times 1$

given $\dot{x}$ find $\dot{q}$

$$W = W^T > 0$$

$$g(\dot{q}) = \frac{1}{2} \dot{q}^T W \dot{q}$$

The problem can be solved with lagrange multipliers.

We modify the cost function as follows.

$$g(\dot{q}, \lambda) = \frac{1}{2} \dot{q}^T W \dot{q} + \lambda^T (\dot{x} - J\dot{q})$$

Lagrange multiplier.

The solution has to satisfy

$$\left(\frac{\partial g}{\partial \dot{q}}\right)^T = 0 \quad (1) \qquad \left(\frac{\partial g}{\partial \lambda}\right)^T = 0 \quad (2)$$

$$(1) \longrightarrow \quad W\dot{q} - J^T \lambda = 0$$

$$\dot{q} = W^{-1} J^T \lambda \qquad (3)$$

(2) $\rightarrow$ $\dot{x} = J\dot{q}$ (4)

(3) & (4) $\rightarrow$ $\dot{x} = JW^{-1}J^{T}d$ (5)

Since the rank$(J) = 6$ then rank$(JW^{-1}J^{T})$ is 6 as well.

$$\det(JW^{-1}J^{T}) \neq 0$$

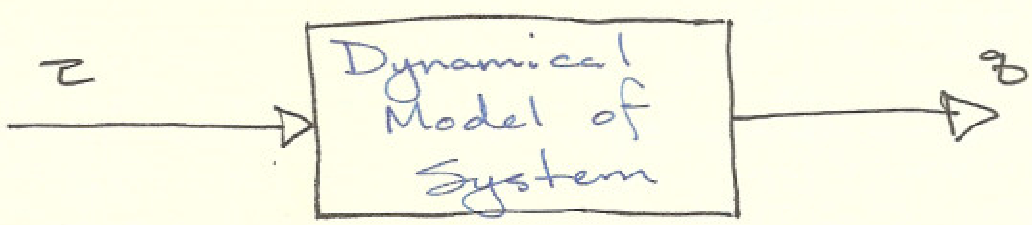
(5) $\rightarrow$ $d = (JW^{-1}J^{T})^{-1}\dot{x}$ (6)

(6) & (3) $\rightarrow$ $\dot{q} = W^{-1}J^{T}(JW^{-1}J^{T})^{-1}\dot{x}$

If $W = I$ then we can find the Pseudo inverse.

$$\dot{q} = \underbrace{J^{T}(JJ^{T})^{-1}}_{\text{Pseudo inverse of } J}\dot{x}$$

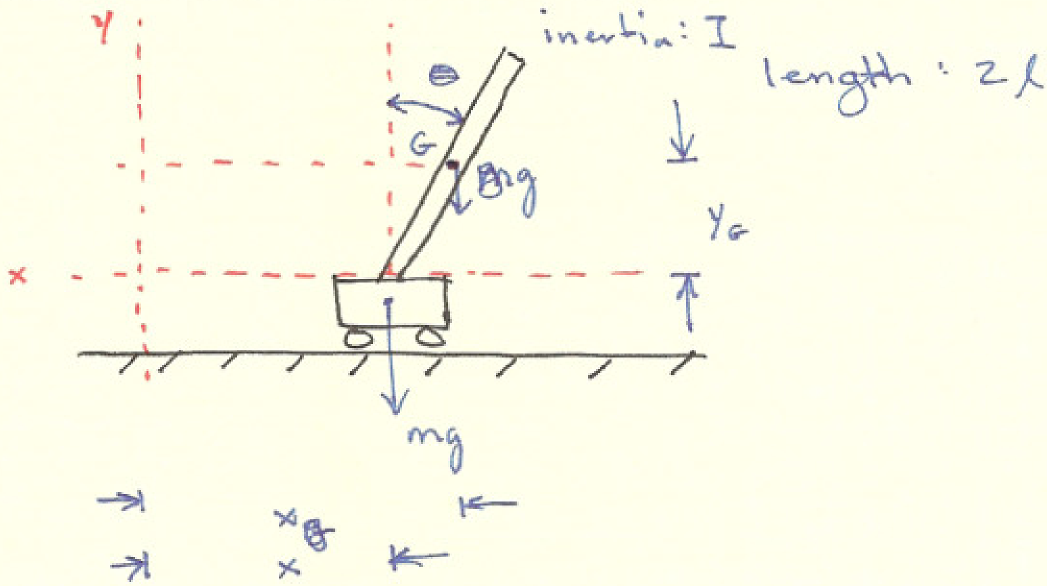
Dynamics



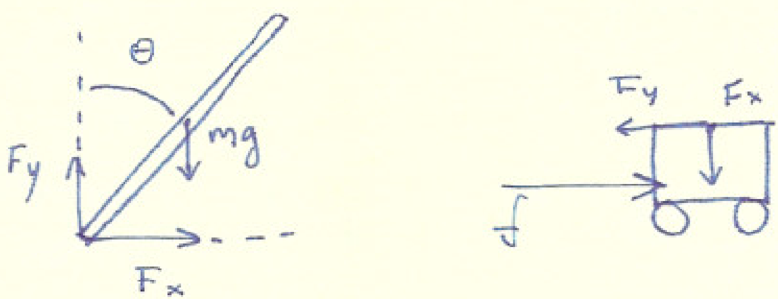
In order to design a controller for a robot manipulator we need to know the relationship between the control inputs (which are generally tourques applied to the manipulator points) and the joint positions, joint velocities and joint accelleration.

EX: Inverted Pendulum. on a cart.

Newton - Euler Method



Typically we will superate all degrees of freedom.



$$x_G = x + l\sin\theta$$

$$\dot{x}_G = \dot{x} + l\dot{\theta}\cos\theta$$

$$\ddot{x}_G = \ddot{x} + l\ddot{\theta}\cos\theta - l\dot{\theta}^2\sin\theta \qquad (1)$$

$$y_G = l\cos\theta$$

$$\dot{y}_G = -l\dot{\theta}\sin\theta$$

$$\ddot{y}_G = -l\ddot{\theta}\sin\theta - l\dot{\theta}^2\cos\theta \qquad (2)$$

$$f - f_x = M\ddot{x} \qquad (3)$$

$$f_x = m\ddot{x}_G \qquad (4)$$

$$f_y - mg = m\ddot{y}_G \qquad (5)$$

$$f_y l\sin\theta - f_x l\cos\theta = I\ddot{\theta} \qquad (6)$$

Solve for f_x & f_y from (4) & (5) and substitude in (6) & (3). Replace $\ddot{x}_G$ and $\ddot{y}_G$ by (1) & (2)
From (3) & (6)

$$\boxed{\begin{aligned} &(M+m)\ddot{x} + m\ddot{\theta}l\cos\theta - m\dot{\theta}^2 l\sin\theta = f. \\ &(I+ml^2)\ddot{\theta} + ml\ddot{x}\cos\theta - mgl\sin\theta = 0 \end{aligned}}$$

Dynamical Model.

Euler – Lagrange Method.

$$L = K - V$$

kinetic energy

potential energy.

Lagrangian.

$$K = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left(\dot{x}_G^{\,2} + \dot{y}_G^{\,2} \right) + \frac{1}{2} I \dot{\theta}^2$$

$$V = mgl \cos\theta$$

$$L = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \left((\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2 \right)$$
$$+ \frac{1}{2} I \dot{\theta}^2 - mgl\cos\theta$$

Now using the Euler-Lagrange equations

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i}$$

F_i : extended force or tourque

n : number of DOF.

i : $1, 2, \ldots n$.

In the case for our system.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = f.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \theta} = -m\ell\dot{x}\dot{\theta}\sin\theta + mg\ell\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m(\dot{x} + \ell\dot{\theta}\cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = (m\ell^2 + I)\dot{\theta} + m\ell\dot{x}\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + m\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (m\ell^2 + I)\ddot{\theta} + m\ell\ddot{x}\cos\theta - m\ell\dot{x}\dot{\theta}\sin\theta$$

$$\boxed{\begin{aligned} (m+M)\ddot{x} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta &= f. \\ (m\ell^2 + I)\ddot{\theta} + m\ell\ddot{x}\cos\theta - mg\ell\sin\theta &= 0 \end{aligned}}$$

Euler - Lagrange Method.

This approach is based on the kinetic and potenial energy to determine the equations of motion. (AKA Dynamical model)

Euler - Lagrange equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i$$

$$L = K - V$$

V: total potential energy.

K: total kinetic energy.

Expressions for Kinetic and Potential Energy.

The Kinetic energy of an object is the sum of two terms: the translational energy obtained by concentrating the entire mass of the object at the center of mass; and the rotation of the object with respect to the center of mass.

The potential energy of the object is the same as that obtained by the concentrating the entire mass of the object at the center of mass.

Once the kinetic and potential energy of each link of the manipulator is known the Lagrangian of the overall manipulator is the sum of the individuals